# Lecture Notes
# Analysis II

For Engineering Students

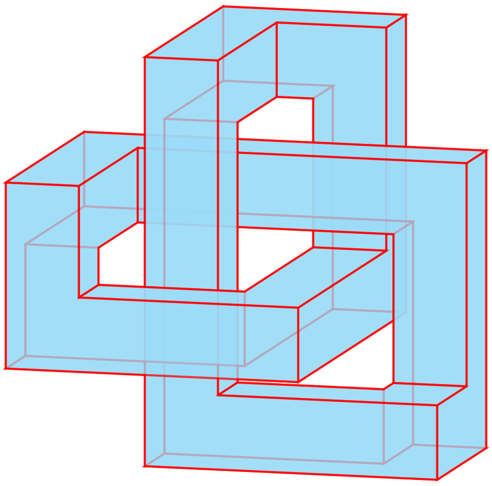
Spring Semester 2025

# Contents

# Chapter 1

# The Euclidean space $\mathbb{R}^n$

In Analysis 1 you have learned the fundamental concepts of differential and integral calculus of real-valued functions in one real variable, known as *Single Variable Calculus*. However, real-life phenomena often depend on a multitude of factors and it requires more than just one variable to properly model such situations. This leads to the study of the theory of differentiation and integration of functions in several variables, called *Multivariable Calculus*. The mathematical stage on which the study of functions in several variables unfolds is the $n$-dimensional Euclidean space $\mathbb{R}^n$.



Before defining the $n$-dimensional Euclidean space and its intrinsic topology, let us recall some basic notions commonly used in analysis and calculus.

- $\mathbb{N}$ the *natural numbers* $\{1, 2, 3, 4, \ldots\}$,
- $\mathbb{Z}$ the *integers*, i.e., signed whole numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- $\mathbb{Q}$ the *rational numbers* $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$,
- $\mathbb{R}$ the *real numbers*,
- $\mathbb{C}$ the *complex numbers*,

An *open interval* is an interval that does not include its boundary points and is denoted by parentheses. The open intervals are thus one of the forms

$$\begin{aligned}(a,b) &= \{x \in \mathbb{R} : a < x < b\},\\ (-\infty,b) &= \{x \in \mathbb{R} : x < b\},\\ (a,+\infty) &= \{x \in \mathbb{R} : a < x\},\\ (-\infty,+\infty) &= \mathbb{R},\end{aligned}$$

where $a$ and $b$ are real numbers with $a \leqslant b$. The interval $(a,a) = \emptyset$ is the empty set, a degenerate interval. Open intervals are *open sets* in the topology of $\mathbb{R}$.

A *closed interval* is an interval that includes all its boundary points and is denoted by square brackets. Closed intervals take the form

$$\begin{aligned}[a,b] &= \{x \in \mathbb{R} : a \leqslant x \leqslant b\},\\ (-\infty,b] &= \{x \in \mathbb{R} : x \leqslant b\},\\ [a,+\infty) &= \{x \in \mathbb{R} : a \leqslant x\},\\ (-\infty,+\infty) &= \mathbb{R},\end{aligned}$$

Closed intervals are *closed sets* in the topology of $\mathbb{R}$. Note that the interval $\mathbb{R} = (-\infty,+\infty)$ is both open and closed at the same time.

A *half-open interval* is a finite interval that includes one endpoint but not the other. It can be left-open or right-open, depending on which endpoint is excluded:

$$\begin{aligned}(a,b] &= \{x \in \mathbb{R} : a < x \leqslant b\},\\ [a,b) &= \{x \in \mathbb{R} : a \leqslant x < b\},\end{aligned}$$

Note that half-open intervals are neither open nor closed sets in the topology of $\mathbb{R}$.

Intervals of the form $[a,b]$, $[a,b)$, $(a,b]$, $(a,b)$ for $a,b \in \mathbb{R}$ with $a \leqslant b$ are called *bounded intervals*, whereas intervals like $(-\infty,b]$, $(-\infty,b)$, $[a,+\infty)$, and $(a,+\infty)$ are *unbounded intervals*.

## 1.1 The vector space $\mathbb{R}^n$

Given a positive integer $n$, the set $\mathbb{R}^n$ is defined as the set of all ordered $n$-tuples $(x_1,\ldots,x_n)$ of real numbers. It is called the *standard Euclidean space of dimension* $n$, or simply the $n$*-dimensional Euclidean space*.

We can represent an element of $\mathbb{R}^n$ either as an $n$-tuple, which is the same as a row vector with $n$ entries,

$$\mathbf{x} = (x_1,\ldots,x_n)$$

or as a column vector with $n$ entries

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

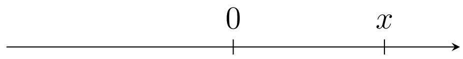
Both representations are common and widely used in the literature. We will generally use column vectors to denote elements of $\mathbb{R}^n$ in calculations, and row vectors to denote elements of $\mathbb{R}^n$ as input parameters of functions defined on $\mathbb{R}^n$.

There are also different ways in which elements in $\mathbb{R}^n$ are denoted, the three most common are

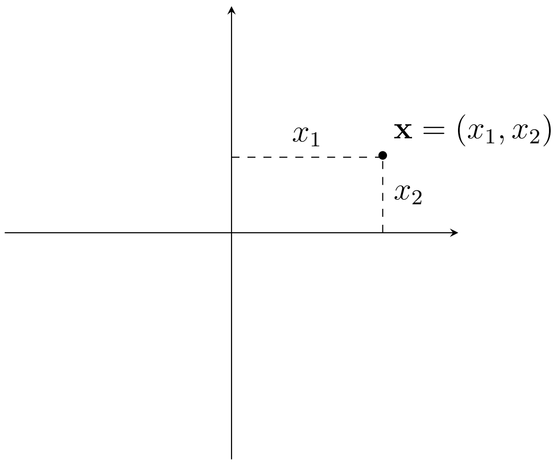
$$x, \qquad \mathbf{x}, \qquad \text{and} \qquad \vec{x}.$$

In this text, we will predominantly use $x$ for elements in $\mathbb{R}$ and $\mathbf{x}$ for elements in $\mathbb{R}^n$ for $n \geqslant 2$.

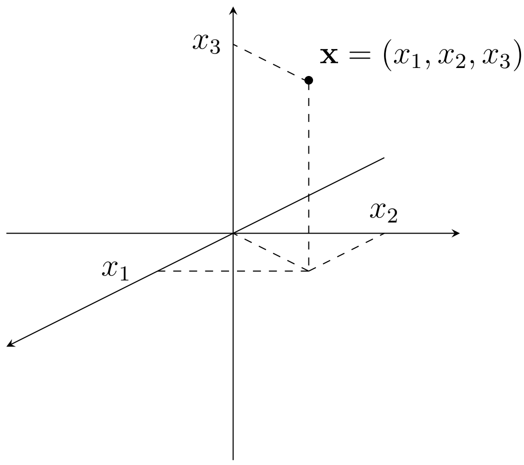
If $n = 1$ then $\mathbb{R}^1 = \mathbb{R}$ corresponds to the real line.




If $n = 2$ then $\mathbb{R}^2$ corresponds to the 2-dimensional plane. A point in $\mathbb{R}^2$ is usually written as either $(x, y)$ or $\mathbf{x} = (x_1, x_2)$.




If $n = 3$ then $\mathbb{R}^3$ corresponds to the 3-dimensional space. A point in $\mathbb{R}^3$ is usually written as eitehr $(x, y, z)$ or $\mathbf{x} = (x_1, x_2, x_3)$.



The set $\mathbb{R}^n$ is an $n$-dimensional inner product vector space over the real numbers. This means it is closed under addition, scalar multiplication, and endowed with an inner product called the scalar product. The addition on $\mathbb{R}^n$ is defined coordinate wise by

$$\mathbf{x}+\mathbf{y}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}+\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}=\begin{pmatrix}x_1+y_1\\ \vdots\\ x_n+y_n\end{pmatrix}.$$

The multiplication of an element $\mathbf{x}\in\mathbb{R}^n$ by a scalar $\lambda\in\mathbb{R}$ is defined as

$$\lambda\mathbf{x}=\lambda\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}\lambda x_1\\ \vdots\\ \lambda x_n\end{pmatrix}.$$

The way in which addition and multiplication on $\mathbb{R}^n$ interact is described by the distributive law, which asserts that

$$\lambda(\mathbf{x}+\mathbf{y})=\lambda\mathbf{x}+\lambda\mathbf{y}. \qquad \text{(Distributive Law)}$$

The vector space $\mathbb{R}^n$ is also equipped with a *scalar product* $\langle\cdot,\cdot\rangle\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ defined as

$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{k=1}^{n}x_k y_k.$$

The scalar product satisfies the three following properties:

1. **Positive-definiteness:** $\langle\mathbf{x},\mathbf{x}\rangle\geqslant 0$ for all $\mathbf{x}\in\mathbb{R}^n$, with equality only for $\mathbf{x}=\mathbf{0}$.
2. **Symmetry:** $\langle\mathbf{x},\mathbf{y}\rangle=\langle\mathbf{y},\mathbf{x}\rangle$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.
3. **Bilinearity:** $\langle\alpha\mathbf{x}+\beta\mathbf{y},\mathbf{z}\rangle=\alpha\langle\mathbf{x},\mathbf{z}\rangle+\beta\langle\mathbf{y},\mathbf{z}\rangle$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^n$ and $\alpha,\beta\in\mathbb{R}$.

In linear algebra, a vector $\mathbf{x}$ is also an $n\times 1$ matrix. Its transpose, written $\mathbf{x}^\top=(x_1,\dots,x_n)$, is therefore a $1\times n$ matrix, and we can interpret the scalar product of two vectors $\mathbf{x},\mathbf{y}$ as the matrix product of $\mathbf{x}^\top$ and $\mathbf{y}$:

$$\langle\mathbf{x},\mathbf{y}\rangle=\mathbf{x}^\top\mathbf{y}=(x_1,\dots,x_n)\cdot\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}.$$

## 1.2 The Euclidean distance on $\mathbb{R}^n$

To be able to extend the analytical methods presented in Analysis 1 to the space $\mathbb{R}^n$, it is important to endow $\mathbb{R}^n$ with a topological structure. On $\mathbb{R}$ we have used the absolute value to define a distance $d(x,y)=|x-y|$, which was then used to define notions such as convergence and continuity in $\mathbb{R}$. We seek to generalize the absolute value and the distance to the space $\mathbb{R}^n$. To do so, we will introduce the concepts of a norm and a metric.

**Definition 1.1** (The Euclidean norm on $\mathbb{R}^n$)**.** The *Euclidean norm* on $\mathbb{R}^n$ is the function $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left( \sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}. \tag{1.1}$$

It measures the distance of the point $\mathbf{x}$ to the origin $\mathbf{0} = (0, \dots, 0)$.

Observe that in one dimension, the Euclidean norm of a real number is the same as the absolute value of that number. In general, the Euclidean norm satisfies the following properties:

1. **Non-negativity:** $\|\mathbf{x}\|_2 \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, with equlity if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogeneity:** $\|\lambda \cdot \mathbf{x}\|_2 = |\lambda| \cdot \|\mathbf{x}\|_2$ for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.
3. **Triangle inequality:** $\|\mathbf{x} + \mathbf{y}\|_2 \leqslant \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

One of the most important properties of the scalar product is the *Cauchy-Schwarz inequality*, which says that

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2 \tag{Cauchy-Schwarz}$$

The Euclidean norm $\|\mathbf{x}\|_2$ also corresponds to the length of a vector $\mathbf{x}$. The scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ measures the angle between the two vectors $\mathbf{x}$ and $\mathbf{y}$: if we designate $\theta$ as the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta. \tag{Angle Formula}$$

In particular if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors, i.e., $\theta = \pm\pi/2$, then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. As a consequence, we obtain the famous *Pythagorean theorem*, which says that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal then

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2. \tag{Pythagoras}$$

With the help of the Euclidean norm we can define a metric on $\mathbb{R}^n$ called the Euclidean distance.

**Definition 1.2** (The Euclidean distance on $\mathbb{R}^n$)**.** The *Euclidean distance* on $\mathbb{R}^n$ is the function $d(.,.) \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}. \tag{1.2}$$

The Euclidean distance captures the natural distance between two points in $\mathbb{R}^n$. It satisfies the following three properties:

1. **Non-negativity:** $d(\mathbf{x}, \mathbf{y}) \geqslant 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with equality only when $\mathbf{x} = \mathbf{y}$.
2. **Symmetry:** $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
3. **Triangle inequality:** $d(\mathbf{x}, \mathbf{y}) \leqslant d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z})$.

## 1.3 The topology on $\mathbb{R}^n$

The Euclidean distance $d(\mathbf{x}, \mathbf{y})$ induces a topology on $\mathbb{R}^n$ which underpins all analytical considerations on $\mathbb{R}^n$. In particular, notions such as continuity, convergence, differentiablility and integrability are all defined in terms of this topology. The building blocks of the topology on $\mathbb{R}^n$ are the so-called open balls.

**Definition 1.3** (Open Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) < r\}$$

is called the *open ball* of radius $r$ centered at $\mathbf{a}$.

Open balls are the mathematical conceptualization of "nearness" and an important use of open balls is to topologically distinguish distinct points: if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{y}$ then we can find a sufficiently small open ball centered at $\mathbf{x}$ and another sufficiently small open ball centered at $\mathbf{y}$ such that these two balls don't touch.

Open balls are instances of open sets. An open set is a set with the property that if $\mathbf{x}$ is a point in the set then all points that are sufficiently near to $\mathbf{x}$ also belong to the set. The mathematically precise definition is as follows:

**Definition 1.4** (Open set)**.** A subset $U \subseteq \mathbb{R}^n$ is *open* if for any point $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that the open ball $B(\mathbf{x}, \varepsilon)$ is contained in $U$.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are open. Also, as was already mentioned, any open ball $B(\mathbf{a}, r)$ is an open set.

**Example 1.1** (Open Sets in $\mathbb{R}^n$)**.**

1. If $a < b$ are real numbers then the interval

   $$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

   is an open set. Indeed, if $x \in (a, b)$, simply take $r = \min\{x - a, b - x\}$. Both these numbers are strictly positive, since $a < x < b$, and so is their minimum. Then the "1-dimensional ball" $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ is a subset of $(a, b)$. This proves that $(a, b)$ is an open set.
2. The infinite intervals $(a, \infty)$ and $(-\infty, b)$ are also open but the intervals

   $$(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\} \qquad \text{and} \qquad [a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

   are not open sets.
3. The rectangle

   $$(a, b) \times (c, d) = \{(x, y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$$

   is an open set.

The antithetical notion to an open set is that of a closed set.

**Definition 1.5** (Closed set)**.** A subset $C \subseteq \mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \backslash C$ is open.

The empty set $\emptyset$ and the space $\mathbb{R}^n$ are the only sets that are both closed and open at the same time. Intuitively, one should think of a closed set as a set that has no "punctures" or "missing endpoints", i.e., it includes all limiting values of points. For instance, the punctured plane $\mathbb{R}^2 \backslash \{(0,0)\}$ is not a closed set.

An example of a closed set is the closed ball.

**Definition 1.6** (Closed Ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$\overline{B(\mathbf{a}, r)} = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) \leqslant r\}$$

is called the closed ball of radius $r$ centered at $\mathbf{a}$. It is a closed set.

**Example 1.2** (Closed Sets in $\mathbb{R}^n$)**.**

1. The closed interval

$$[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

   is a closed set, because its complement $\mathbb{R} \backslash [a, b] = (-\infty, a) \cup (b, \infty)$ is an open set.
2. Infinite intervals with closed boundary $[a, \infty)$ and $(-\infty, b]$ are closed sets.
3. Halfopen intervals such as $[a, b)$ or $(a, b]$ are neither closed nor open sets.
4. Any set consisting of only finitely many points is a closed set.

The following two propositions describe how open and closed sets behave under basic set manipulations such as unions, intersections, and set differences.

**Proposition 1.1.**

- *If $U \subseteq \mathbb{R}^n$ is open and $C \subseteq \mathbb{R}^n$ is closed then $U \backslash C$ is open.*
- *If $C \subseteq \mathbb{R}^n$ is closed and $U \subseteq \mathbb{R}^n$ is open then $C \backslash U$ is closed.*

**Proposition 1.2.**

- *If $U_1, \ldots, U_k \subseteq \mathbb{R}^n$ are open then $U_1 \cup \ldots \cup U_k$ and $U_1 \cap \ldots \cap U_k$ are open.*
- *If $C_1, \ldots, C_k \subseteq \mathbb{R}^n$ are closed then $C_1 \cup \ldots \cup C_k$ and $C_1 \cap \ldots \cap C_k$ are closed.*

To better grasp the difference between open sets and closed sets, we introduce the concept of interior points, exterior points, and boundary points.

**Definition 1.7** (Interior, Exterior, Boundary Points)**.** Let $S$ be a subset of $\mathbb{R}^n$ and $\mathbf{x}$ a point in $\mathbb{R}^n$.

(i) We call $\mathbf{x}$ an *interior point* of $S$ if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ is contained in $S$.

(ii) We call $\mathbf{x}$ an *exterior point* of $S$ if there exists $r > 0$ such that the ball $B(\mathbf{x}, r)$ has empty intersection with $S$.

(iii) We call $\mathbf{x}$ a *boundary point* of $S$ if it is neither an interior point nor an exterior point for $S$. Equivalently, $\mathbf{x}$ is a boundary point of $S$ if for every $r > 0$ the ball $B(\mathbf{x}, r)$ has non-empty intersection with $S$ without being entirely contained in $S$.

Note that every point is either interior, exterior or on the boundary in relationship to a set $S$.

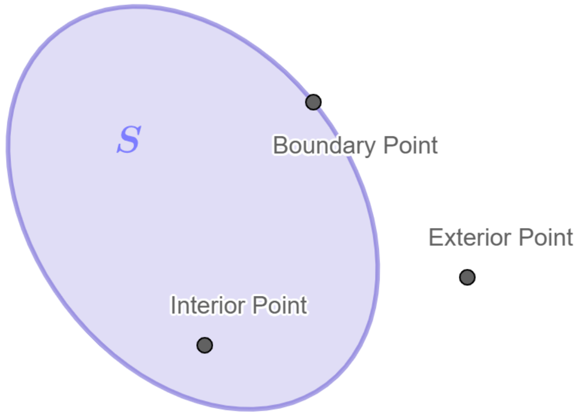



Figure 1.1: Illustration of the difference between interior, exterior and boundary points of a set $S$.

**Definition 1.8** (Interior)**.** The set of all interior points of a set $S$ is called the interior of $S$ and it is denoted by $\mathring{S}$.

**Definition 1.9** (Boundary)**.** The set of all boundary points of a set $S$ is called the boundary of $S$ and we use $\partial S$ to denote it.

**Definition 1.10** (Closure)**.** The closure of $S$, denoted by $\overline{S}$, is the set of points $\mathbf{x} \in \mathbb{R}^n$ with the property that for all $r > 0$ one has

$$B(\mathbf{x}, r) \cap S \neq \emptyset.$$

Equivalently, the closure of $S$ is the union of all its interior points and all its boundary points.

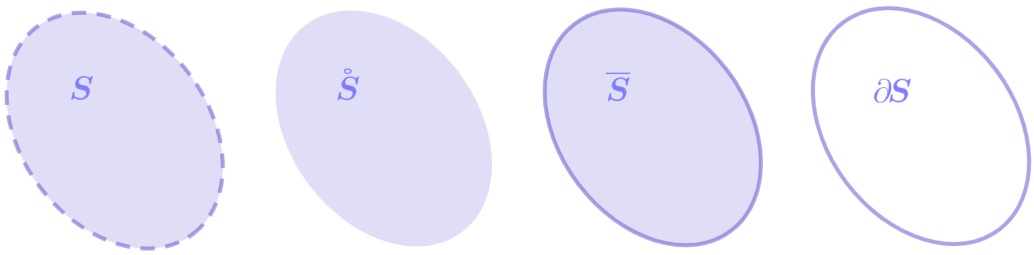



Figure 1.2: The interior, closure and boundary sets of a set $S$.

Clearly, we have the set inclusions $\mathring{S} \subseteq S \subseteq \overline{S}$. To summarize, the closure of $S$ is $S$ plus its boundary, its interior is $S$ minus its boundary, and the boundary is the closure minus the interior:

$$\mathring{S} = S \backslash \partial S \qquad \overline{S} = S \cup \partial S, \qquad \text{and} \qquad \partial S = \overline{S} \backslash \mathring{S}.$$

**Proposition 1.3.** *Let $S \subseteq \mathbb{R}^n$. The interior $\mathring{S}$ is the largest open set contained inside of $S$. The closure $\overline{S}$ is the smallest closed set that has $S$ as a subset.*

**Corollary 1.1.** *A set is open if and only if it is equal to its interior. On the other hand, a set is closed if and only if it is equal to its closure, which is the same as saying that it contains all its boundary points.*

**Example 1.3** (Closure, Interior, Boundary)**.**

1. The sets $(0,1)$, $[0,1]$, $[0,1)$, and $(0,1]$ all have the same closure, interior, and boundary: the closure is $[0,1]$, the interior is $(0,1)$, and the boundary consists of the two points 0 and 1.
2. The sets

$$\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\} \qquad \text{and} \qquad \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 1\}$$

   both have the same closure, interior, and boundary: the closure is the disc of equation $x^2+y^2\leqslant 1$, the interior is the disc of equation $x^2+y^2<1$, and the boundary is the circle of equation $x^2+y^2=1$.
3. The set

$$U=\{(x,y)\in\mathbb{R}^2 : |y|<x^2\}$$

   describes the region between two parabolas touching at the origin, shown in Fig. 1.3. The set is open, so $U=\mathring{U}$. The closure of $U$ is given by

$$\overline{U}=\{(x,y)\in\mathbb{R}^2 : |y|\leqslant x^2\}.$$

   In particular, the closure contains the point $(0,0)$.

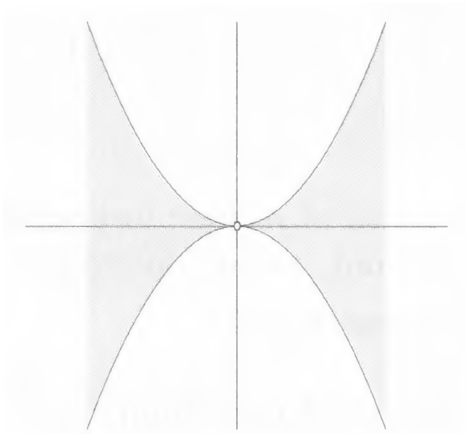


Figure 1.3: The origin belongs to the closure of the shaded region.

4. The unit ball is open in $\mathbb{R}^n$ and is defined by

$$B_1=B(\mathbf{0},1)=\{\mathbf{x}\in\mathbb{R}^n : \|\mathbf{x}\|_2<1\}$$

   Its boundary is the sphere $\partial B_1=\{\mathbf{x}\in\mathbb{R}^n : \|\mathbf{x}\|_2=1\}$.
5. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function. The set

$$G_f=\{(x,f(x))\in\mathbb{R}^2 : x\in\mathbb{R}\}$$

is known as the graph of $f$ and represents a curve in $\mathbb{R}^2$. We have $\mathring{G}_f = \emptyset$. Therefore $G_f = \partial G_f$. The closed graph theorem says that graph $\mathring{G}_f$ is a closed set in $\mathbb{R}^2$ if $f$ is a continuous function.

6. Let $B = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$ and $I = [0, 5]$. The set $S$ defined by

$$S = B \times I = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\}$$

is a cylinder. The set $S$ is neither closed nor open. The boundary of $S$ is given by

$$\partial S = \underbrace{\partial B \times I}_{E_1} \cup \underbrace{B \times \partial I}_{E_2},$$

where

$$E_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1 \text{ and } 0 \leqslant x_3 \leqslant 5\right\},$$
$$E_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 \text{ and } x_3 \in \{0, 5\}\right\}.$$

**Definition 1.11** (Neighborhood of a point in $\mathbb{R}^n$)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$. If $\mathbf{x}$ is an interior point of $U$ then $U$ is called a *neighborhood of* $\mathbf{x}$.

## 1.4 Sequences in $\mathbb{R}^n$

Limits of sequences and limits of functions are fundamental notions in calculus, as you already have seen in Analysis 1. Let us extend these principles to higher dimensions. We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of natural numbers.

**Definition 1.12** (Sequences in $\mathbb{R}^n$)**.** A *sequence* of elements of $\mathbb{R}^n$ is a function $k \mapsto \mathbf{x}_k$ that associates to every natural number $k \in \mathbb{N}$ an element $\mathbf{x}_k \in \mathbb{R}^n$. We write $(\mathbf{x}_k)_{k\in\mathbb{N}}$ to denote a sequence in $\mathbb{R}^n$.

Although $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is by definition a sequence of $n$-tuples, we can also think of it as an $n$-tuple of sequences by considering each coordinate as an individual sequence,

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix}.$$

**Definition 1.13** (Convergent sequence)**.** A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{x} \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists $N > 1$ such that when $k \geqslant N$, then $d(\mathbf{x}_k, \mathbf{x}) < \varepsilon$. In this case we call $\mathbf{x}$ the *limit* of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and write

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

Note that not every sequence has a limit, but if a sequence does then this limit is unique. Sequences that possess a limit are called *convergent*, whereas sequences that don't possess one are called *divergent*.

It follows from Definition 1.13 that a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if the sequence of distances $d(\mathbf{x}_k, \mathbf{x})$ converges to 0, i.e.,

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} d(\mathbf{x}_k, \mathbf{x}) = 0.$$

Convergence is also observed coordinate wise: A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ if and only if each coordinate of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to the respective coordinate of $\mathbf{x}$. More precisely, if

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \quad \Longleftrightarrow \quad \lim_{k\to+\infty} x_{i,k} = x_i \;\text{ for all } i = 1, \ldots, n.$$

**Example 1.4** (Convergent and divergent sequences in $\mathbb{R}^n$).

1. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} e^{-k} \\ \frac{k}{k+1} \\ \frac{1}{\sqrt{k^2-k}-k} \end{pmatrix}$$

   converges as $k \to +\infty$ to the limit

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

   because $\lim_{k\to+\infty} e^{-k} = 0$, $\lim_{k\to+\infty} \frac{k}{k+1} = 1$, and $\lim_{k\to+\infty} \frac{1}{\sqrt{k^2-k}-k} = -2$.

2. The sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{x}_k = \begin{pmatrix} 0 \\ \frac{1-(-1)^k}{2} \end{pmatrix}$$

   diverges because it diverges in the second coordinate.

The following properties describe the arithmetic operations of sequences in the $n$-dimensional Euclidean space and tell us that limits cooperate nicely with the vector space structure of $\mathbb{R}^n$.

**Properties of limits of sequences.** Let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}^n$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

1. **Addition of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\mathbf{x}_k + \mathbf{y}_k)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \mathbf{x}_k + \mathbf{y}_k = \lim_{k\to+\infty} \mathbf{x}_k + \lim_{k\to+\infty} \mathbf{y}_k.$$

2. **Multiplication of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ both converge then so

does $(\lambda_k \mathbf{x}_k)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \lambda_k \mathbf{x}_k = \left(\lim_{k\to+\infty} \lambda_k\right) \cdot \left(\lim_{k\to+\infty} \mathbf{x}_k\right).$$

3. **Inner product of sequences:** If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}}$ both converge then so does $(\langle \mathbf{x}_k, \mathbf{y}_k\rangle)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \langle \mathbf{x}_k, \mathbf{y}_k\rangle = \left\langle \lim_{k\to+\infty} \mathbf{x}_k, \lim_{k\to+\infty} \mathbf{y}_k \right\rangle.$$

**Definition 1.14** (Cauchy sequences)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N > 1$ such that $k, l \geqslant N$ implies $d(\mathbf{x}_k, \mathbf{x}_l) < \varepsilon$.

**Theorem 1.1.** *Every convergent sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence and every Cauchy sequence is convergent.*

**Proposition 1.4.** *Let $S \subseteq \mathbb{R}^n$ be a non-empty set and suppose $\mathbf{x} \in \partial S$ is a boundary point of $S$. Then there exists a sequence of elements in $\mathring{S}$, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \in \mathring{S}$, such that*

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}.$$

The following example provides an illustration of the content of Proposition 1.4.

**Example 1.5.** Consider the open ball of radius 5 centered at the origin in $\mathbb{R}^2$,

$$B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 25\}.$$

The boundary of $B((0,0),5)$ is the circle of radius 5 centered at the origin, i.e.,

$$\partial B(\mathbf{0}, 5) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 5\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 25\}.$$

Any point $\mathbf{x} \in \partial B(\mathbf{0}, 5)$ of this circle takes the form

$$\mathbf{x} = \begin{pmatrix} 5\cos\theta \\ 5\sin\theta \end{pmatrix}, \quad \text{for some } \theta \in [0, 2\pi).$$

We can define a sequence

$$\mathbf{x}_k = \begin{pmatrix} \frac{5k}{k+1}\cos\theta \\ \frac{5k}{k+1}\sin\theta \end{pmatrix},$$

and note that $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$. So we see that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence of points inside the open ball $B(\mathbf{0}, 5)$ converging to the point $\mathbf{x}$ on the border .

**Proposition 1.5.** *Let $S \subseteq \mathbb{R}^n$ be a non-empty subset of $\mathbb{R}^n$ and let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ be a sequence of elements in $S$. If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges then the limit $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x}$ must belong to $\overline{S}$, the closure of $S$.*

**Example 1.6.** Consider the "halfopen" rectangle

$$S = [0, 1] \times [0, 1).$$

This is not a closed set, because the point $(\frac{2}{3}, 1)$, for example, is in the boundary $\partial S$ but not in $S$ itself. Moreover, the sequence

$$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{3}{4} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \dots$$

is a sequence of points in the interior of $S$ that converge to the point $(\frac{2}{3}, 1)$, which is not part of $S$, but it is part of the closure of $S$.

**Definition 1.15** (Bounded set)**.** A subset $E \subseteq \mathbb{R}^n$ is *bounded* if it is contained in a ball of finite radius centered at the origin:

$$E \subseteq B(\mathbf{0}, R) \quad \text{for some } R < \infty.$$

Note that a closed set need not be bounded. For instance, the interval $[0, \infty)$ is closed, but it is not a bounded.

**Definition 1.16** (Compact set)**.** A subset $C \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Compactness is the basic "finiteness criterion" for subsets of $\mathbb{R}^n$. An important characterization of compact sets in Euclidean spaces is given by the Bolzano-Weierstrass theorem. Before we can state this theorem, we need to recall what is a subsequence.

**Definition 1.17** (Subsequence)**.** A *subsequence* of a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is any sequence of the form $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$, where $(k_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of positive integers.

If a sequence converges then any subsequence of it also converges to the same limit.

**Theorem 1.2** (Bolzano-Weierstrass theorem in $\mathbb{R}^n$)**.** *Let $C \subseteq \mathbb{R}^n$ be compact. Any sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of elements in $C$ possesses a convergent subsequence $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$ whose limit is in $C$.*

**Definition 1.18** (Bounded sequences in $\mathbb{R}^n$)**.** A sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ is *bounded* if there exists a constant $C > 0$ such that $\|\mathbf{x}_k\|_2 \leqslant C$ for any $k \in \mathbb{N}$.

Note that every convergent sequence is a bounded sequence, but the opposite is in general not true. For example, the sequence $x_k = (-1)^k$ is bounded and does not converge. The following is an immediate corollary of the Bolzano-Weierstrass theorem.

**Corollary 1.2.** *Each bounded sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ has a convergent subsequence $(\mathbf{x}_{k_i})_{i\in\mathbb{N}}$.*

# Chapter 2

# Real-valued functions in $\mathbb{R}^n$

*Multivariable calculus*, also known as *multivariate calculus*, is the extension of calculus in one variable to calculus with functions of several variables. We start by defining real-valued functions in more than one variable.

## 2.1 Definition

**Definition 2.1** (Real-valued function on $E \subseteq \mathbb{R}^n$)**.** Let $E$ be a non-empty subset of $\mathbb{R}^n$. A function $f\colon E \to \mathbb{R}$ that assigns to every element $\mathbf{x} \in E$ a unique real number $y = f(\mathbf{x})$ is called a *real-valued function* on $E$.

Given a function $f\colon E \to \mathbb{R}$, the *domain* of $f$ is $E$, denoted $\operatorname{dom}(f)$ or $\operatorname{dom} f$. In theory, the domain should always be a part of the definition of the function rather than a property of it, but in practice it is often the case that the domain is inferred by the description of the function (see Examples 2.1 and 2.3 below).

The *image* (sometimes also called the *range*) of a function $f$ is the set of all the output values that $f$ produces. We denote it by $\operatorname{Im}(f)$ and it is formally defined as

$$\operatorname{Im}(f) = \{f(\mathbf{x}) : \mathbf{x} \in E\} = \{y \in \mathbb{R} : \exists \mathbf{x} \in E \text{ with } f(\mathbf{x}) = y\}.$$

**Example 2.1.** Let us find and sketch the domain of the function

$$f(x, y) = \frac{\sqrt{x + y + 1}}{(x - 1)}.$$

The expression for $f$ makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of $f$ is:

$$\operatorname{dom}(f) = \{(x, y) : x + y + 1 \geqslant 0, x \neq 1\}.$$

The inequality $x + y + 1 > 0$, or $y > -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. See Fig. 2.1 for a sketch of this region.

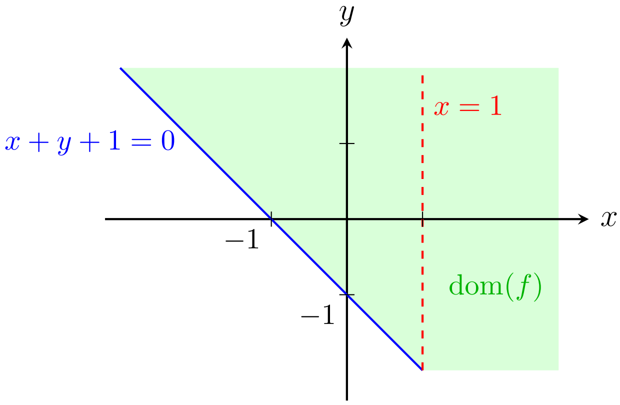



Figure 2.1: The domain of the function $f(x, y) = \frac{\sqrt{x+y+1}}{(x-1)}$
.

The relationship between the domain and the image of a function is described by its *graph*. We use $G(f)$ to denote the graph of a function $f : E \to \mathbb{R}$ and it is given by

$$G(f) = \left\{ \begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix} : \mathbf{x} \in D \right\} \subseteq \mathbb{R}^{n+1}.$$

Note that the graph of $f$ is a subset of $\mathbb{R}^{n+1}$. More precisely, the graph is the hypersurface in $\mathbb{R}^{n+1}$ corresponding to the set of all points $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ that satisfy the equation

$$x_{n+1} = f(x_1, \ldots, x_n).$$

**Example 2.2.** Consider the equation $x + y = z$; as you learned in linear algebra, the solutions to this equation describe a plane in $\mathbb{R}^3$. Now, compare this with the function $f(x, y) = x + y$, a real-valued function in two variables. By definition, the graph of $f(x, y)$ consists of points $(x, y, z) \in \mathbb{R}^3$ where $z = f(x, y)$. For $f(x, y) = x + y$, this gives the equation of the plane $x + y = z$. Thus, the graph of $f(x, y) = x + y$ is exactly the plane $x + y = z$.

Example 2.2 connects what you studied in linear algebra, where you worked with linear equations like $x + y = z$, to what you're learning now in multivariable calculus. But there's more! With multivariable functions, you can describe not just planes, but much more complex geometric surfaces, as this next example illustrates.

**Example 2.3.** Consider the real-valued function $f(x, y) = \sqrt{1 - x^2 - y^2}$, which is a function in 2 variables. The natural domain of this function is $\mathrm{dom}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$, which is the closed disc of radius 1 centered at the origin. The image of $f$ is $\mathrm{Im}(f) = [0, 1]$ and the graph $G(f) = \{(x, y, z) \in D \times \mathbb{R}, z = f(x, y)\}$ coincides with the set of solutions to the equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z \geqslant 0.$$

In other words, the graph of the function is a **semi-sphere**, see Fig. 2.2 below.

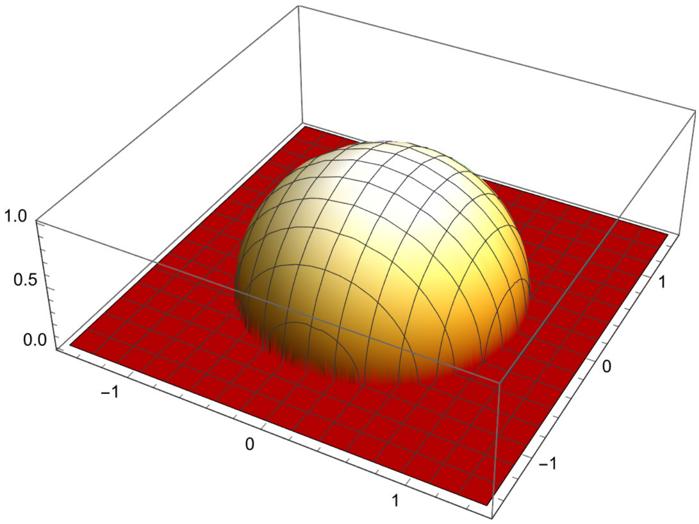



Figure 2.2: Graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$.

**Example 2.4.** In physics, the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ are often called scalar fields. The gravitational potential of a mass or the electric potential of an electric charge are examples of scalar fields:

$$\phi : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}, \quad \phi(\mathbf{x}) = \frac{k}{\|\mathbf{x}\|_2}$$

for a real constant $k$. In mechanics, we often consider systems where the energy is conserved (Hamiltonian systems). For the movement of a particle of mass $m$ in space, subject to the potential $V(\mathbf{x})$, its energy is a real-valued function of its momentum $\mathbf{p} = m\mathbf{v}$ here $\mathbf{v}$ is the velocity and $\mathbf{x}$ the position in space:

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad E(\mathbf{p},\mathbf{x}) = \frac{\|\mathbf{p}\|_2^2}{2m} + V(\mathbf{x}).$$

The movement follows the lines at which the energy $E$ is constant. These lines are called "contour lines" and they are special cases of so-called *level sets*, which we define and discuss next.

## 2.2 Level Sets

**Definition 2.2** (Level set)**.** Let $f\colon E \to \mathbb{R}, E \subseteq \mathbb{R}^n (E \neq \emptyset)$. Given a real number $c \in \mathrm{Im}(f)$, we call the set

$$L_c(f) = \{\mathbf{x} \in D : f(\mathbf{x}) = c\} = f^{-1}(\{c\})$$

the *level set* of $f$ at height $c$. If $c \notin Im(f)$, then $L_c(f) = \emptyset$.

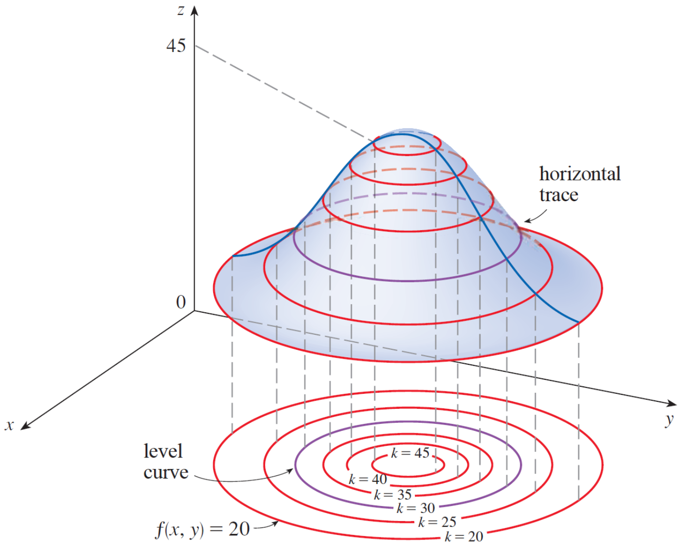



Figure 2.3: The figure displays the graph of a function in 2 variables together with an illustration of its level curves in the $xy$-plane. One can also think of level curves as the projection of the horizontal traces onto the xy-plane, where a *horizontal trace* is a line formed by intersecting the graph of the function with a plane parallel to the xy-plane.

Level sets of functions in 2 variables $f\colon \mathbb{R}^2 \to \mathbb{R}$ are sometimes also called *level curves* (or *contour lines*). It represents all the points where $f$ has *"height" c*. A collection of contour lines is called a *contour map*. Contour maps are very helpful for visualizing functions, and they are most descriptive if the level curves are drawn for equally spaced heights, see Fig. 2.4.

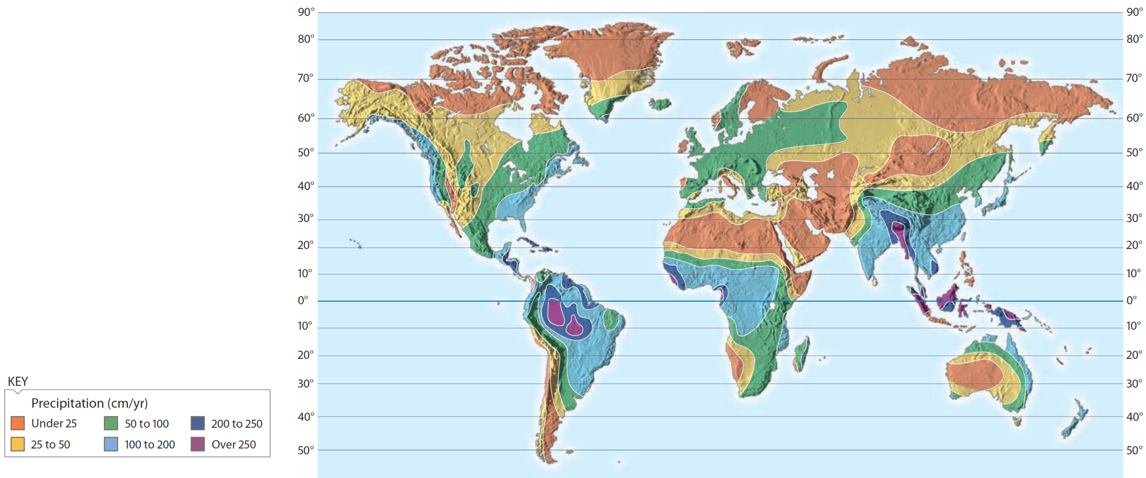



Figure 2.4: Contour map of participation as a function in two variables, the longitude and latitude coordinates on earth.

In summary, we now have learned of two ways of graphically representing a real-valued functions in two variables. The first way is by its graph, which is a hypersurface in $\mathbb{R}^3$, and the second is by a contour map, the projection of its contour lines onto the plane $\mathbb{R}^2$. In Fig. 2.5 below you can see these two methods juxtaposed.

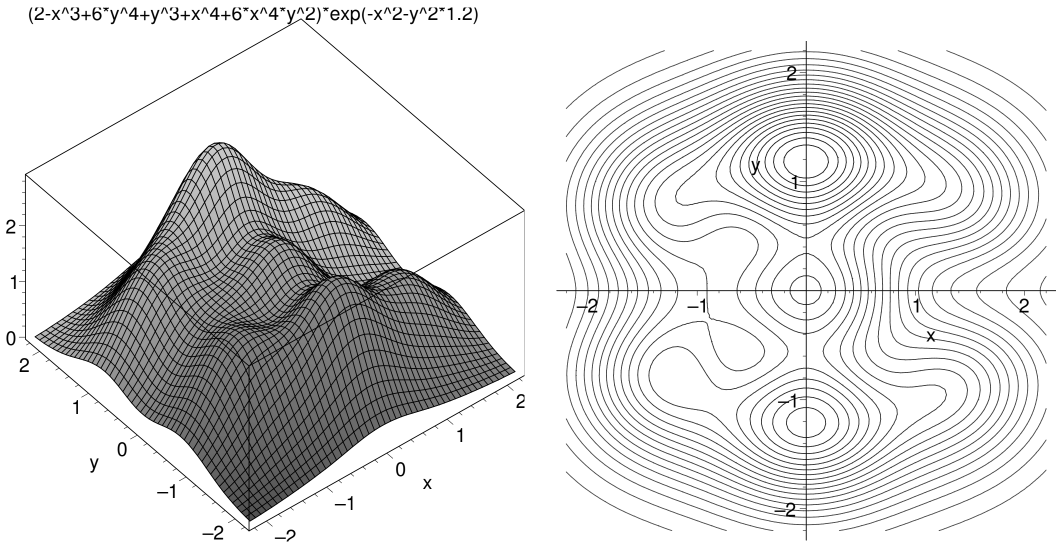



Figure 2.5: Depiction of graph (left) and contour diagram (right) of the same function in 2 variables.

**Example 2.5.** Consider the same function as in Example 2.3, that is, $f(x, y) = \sqrt{1 - x^2 - y^2}$, and let us try to produce a simple contour diagram for it. By definition, the level curve of $f$ at height $c$ is $L_c(f) = \{(x, y) : \sqrt{1 - x^2 - y^2} = c\} \subseteq \mathbb{R}^2$. In Fig. 2.6 we see the level curves of this function at heights $c = 0, \frac{1}{2}, 1$, denoted by $L_0(f)$, $L_{\frac{1}{2}}(f)$, and $L_1(f)$.

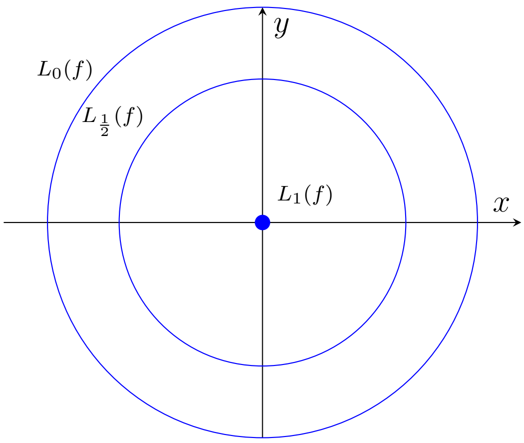



Figure 2.6: Level curves of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ at heights $c = 0$, $c = \frac{1}{2}$, and $c = 1$.

**Example 2.6.** Let $f(x, y) = \frac{xy-1}{\sqrt{y-x^2}}$, whose domain is $\text{dom}(f) = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$. Notice that $\text{dom}(f)$ is open and unbounded.

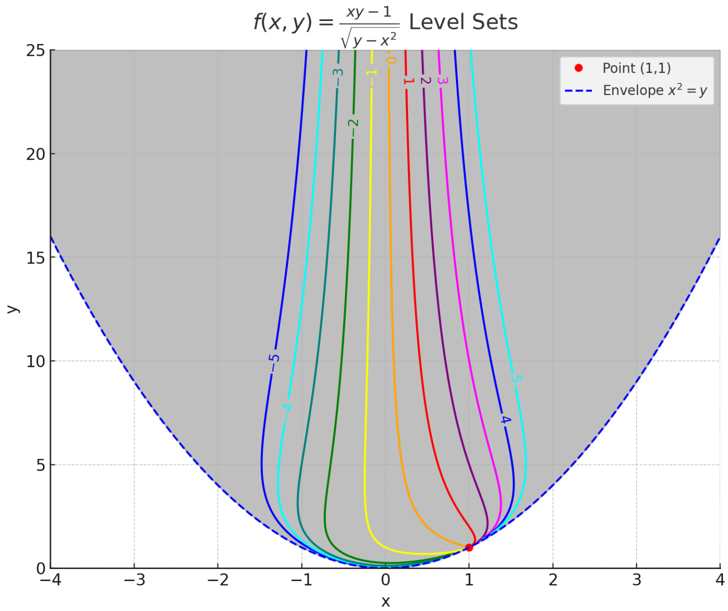



Figure 2.7: The figure displays a series of level curves for the function $f(x,y) = \frac{xy-1}{\sqrt{y-x^2}}$ passing through the point $(1,1)$. As we will explore subsequently, this indicates that the limit of $f(x,y)$ as $(x,y)$ approaches (1,1) is not well-defined.

## 2.3 Limits

**Definition 2.3** (Bounded function)**.** A function $f\colon E \to \mathbb{R}$ is bounded if there exists a number $M \in [0,\infty)$, such that $|f(\mathbf{x})| \leqslant M$ for all $\mathbf{x} \in E$. In this context, we call $M$ an *upper bound of* $f$, or say that $f$ is *bounded by* $M$.

**Definition 2.4.** Let $f\colon E \to \mathbb{R}$ with $E \subseteq \mathbb{R}^n$. We say that $f$ is *defined in a neighborhood of* $\mathbf{x}_0 \in \mathbb{R}^n$ if $\mathbf{x}_0$ is an interior point of $E \cup \{\mathbf{x}_0\}$. That is, there exists $\delta > 0$ such that $B(\mathbf{x}_0, \delta) \subseteq E \cup \{\mathbf{x}_0\}$.

In the above definition, it is possible that $\mathbf{x}_0 \notin E$. In other words, it is possible for a function to be defined in a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^n$ without being defined at $\mathbf{x}_0$ itself.

**Example 2.7.** Consider the function $f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}$ whose domain equals $\operatorname{dom}(f) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \neq 0\} = \mathbb{R}^n \backslash \{\mathbf{0}\}$. Although this function is not defined at $\mathbf{0}$, it is defined in a neighborhood of $\mathbf{0}$.

We are concerned with points where the function is defined in a neighborhood around the point, because this is necessary to properly define the limit of a function at that point. If the function is not defined in the neighborhood of a point, then it is not always possible to talk about the limit of the function at that point without running into mathematical contradictions.

**Definition 2.5** (Limit of a function)**.** Let $E$ be a subset of $\mathbb{R}^n$, $f\colon E \to \mathbb{R}$ a function with domain $E$ and assume $f$ is defined in a neighborhood of the point $\mathbf{x}_0 \in \mathbb{R}^n$. We say that $f$ has a *limit* $l \in \mathbb{R}$ at $\mathbf{x}_0$ and write

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x} \in E$,

$$0 < \|\mathbf{x} - \mathbf{x}_0\| \leqslant \delta \implies |f(\mathbf{x}) - l| \leqslant \varepsilon$$

Note that the limit of a function at a point does not always exist. But if it does exists then it is unique, which means that a function has at most one limit at a given point.

**Example 2.8.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let's calculate its limit as $(x,y)$ approaches $(0,0)$. We will learn several different methods of finding the limit of a function at a point (see, for example, the Squeeze Theorem below), but the most standard method consists of simply verifying Definition 2.5. Given the relation $0 \leqslant \sqrt{x^2+y^2}$, we have

$$\begin{aligned} |f(x,y)| &= \frac{|x+y|\,|x^2-xy+y^2|}{x^2+y^2} \leqslant (|x|+|y|)\frac{x^2+|x||y|+y^2}{x^2+y^2} \\ &\leqslant (|x|+|y|)\frac{x^2+|x||y|+y^2+\frac{1}{2}(|x|-|y|)^2}{x^2+y^2} \\ &= (|x|+|y|)\frac{\frac{3}{2}x^2+\frac{3}{2}y^2}{x^2+y^2} \\ &\leqslant 2\sqrt{x^2+y^2}\frac{\frac{3}{2}x^2+\frac{3}{2}y^2}{x^2+y^2} = 3\sqrt{x^2+y^2} = 3\|(x,y)\|_2. \end{aligned}$$

This shows that as long as $\delta < \frac{\varepsilon}{3}$ we have $\|(x,y)\|_2 < \delta \implies |f(x,y)| \leqslant \varepsilon$. According to Definition 2.5, this means exactly that $\lim_{\mathbf{x}\to(0,0)} f(\mathbf{x}) = 0$.

**Proposition 2.1** (Characterisation of limits by sequences)**.** *Let $E \subseteq \mathbb{R}^n, \mathbf{x}_0 \in \mathbb{R}^n$ and assume $f\colon E \to \mathbb{R}$ defined on a neighbourhood of $\mathbf{x_0}$, and $l \in \mathbb{R}^n$. The following statements are equivalent:*

1. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = l$.
2. $\lim_{n\to\infty} f(a_n) = l$ *for every sequence* $(a_k)_{k\in\mathbb{N}}$ *in* $E\backslash\{\mathbf{x}_0\}$ *converging to* $\mathbf{x}_0$.

**Properties of limits of functions.** Assume $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$ exist.

1. **Linear combinations:** For constants $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (\alpha f(\mathbf{x}) + \beta g(\mathbf{x})) = \alpha\Big(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\Big) + \beta\Big(\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})\Big)$$

2. **Products:**

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (f(\mathbf{x})\cdot g(\mathbf{x})) = \Big(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\Big)\cdot\Big(\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})\Big).$$

3. **Quotients:** If $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) \neq 0$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \left(\frac{f(\mathbf{x})}{g(\mathbf{x})}\right) = \frac{\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})}.$$

4. **Compositions:** Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ be given. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists, and $g_i : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x\to b_i} g_i(x) = a_i$ for each $i$, then

$$\lim_{\mathbf{x}\to\mathbf{b}} f(g_1(x_1), g_2(x_2), \dots, g_n(x_n)) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$

**Example 2.9.** Let us calculate

$$\lim_{(x,y)\to(-3,4)} \frac{1+xy}{1-xy}.$$

Since $\lim_{(x,y)\to(-3,4)} x = -3$ and $\lim_{(x,y)\to(-3,4)} y = 4$, it follows from properties 1 and 2 of limits of functions that

$$\lim_{(x,y)\to(-3,4)} 1+xy \;=\; 1 + (\lim_{(x,y)\to(-3,4)} x)(\lim_{(x,y)\to(-3,4)} y) \;=\; 1+(-3)\cdot 4 = -11.$$

Similarly, we obtain $\lim_{(x,y)\to(-3,4)} 1-xy = 13$. Since the limit of the numerator and denominator exist and the denominator does not converge to 0, it follows from property 3 of limits of functions that

$$\lim_{(x,y)\to(-3,4)} \frac{1+xy}{1-xy} \;=\; \frac{\lim_{(x,y)\to(-3,4)} 1+xy}{\lim_{(x,y)\to(-3,4)} 1-xy} \;=\; -\frac{11}{13}.$$

**Example 2.10.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The graph of this function is depicted in Fig. 2.8 below. What is the limit of this function as $(x,y)$ approaches $(0,0)$? To answer this question, note that $\lim_{t\to 0} f(t,t) = \frac{1}{2}$, whereas $\lim_{t\to 0} f(t,0) = 0$. Since these two limits are different from one another, it follows from property 4 of limits of functions that the limit as $(x,y)$ approaches $(0,0)$ does not exist. This can also be observed graphically. The graph of $f$ is shown in Fig. 2.8 and contains the two lines $\{(x,0,0) : x \in \mathbb{R}\}$ and $\{(0,y,0) : y \in \mathbb{R}\}$, as well as the half-lines $\{(t,t,\frac{1}{2}) : t \in (0,+\infty)\}$ and $\{(t,t,\frac{1}{2}) : t \in (-\infty,0)\}$.

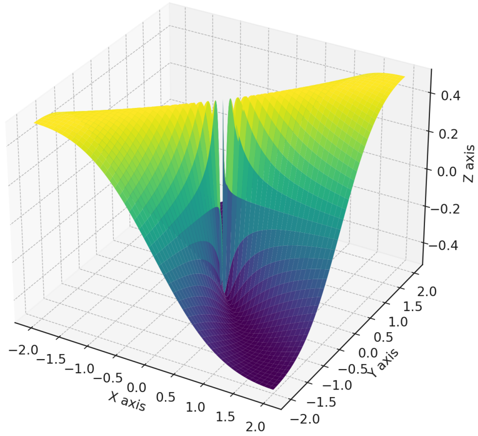



Figure 2.8: Graph of the function $f(x,y) = \frac{xy}{x^2+y^2}$.

**Example 2.11** (The problem with limits in several variables)**.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function in two variables; we would like to determine the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

A (naïve) idea is to compute the two iterated limits:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \qquad \text{or} \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

If these two limits exist and coincide, one might then be led to believe that the limit of the function at $(0,0)$ is equal to 0. However, this is note true! For example, consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

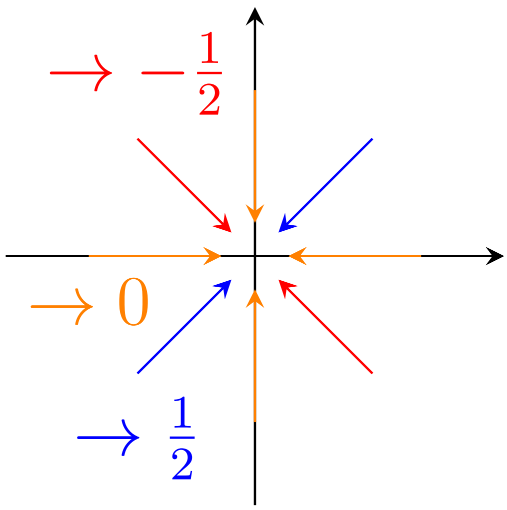
For this particular function, we find that the iterated limits are:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0} = 0,$$
$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = 0.$$

However, instead having the two variables approach 0 one after the other, we can have them approach zero simultaneously, for example along the diagonal $x = y$. In this case, setting both $x$ and $y$ equal to $t$ and letting $t$ go to zero, we obtain

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2},$$

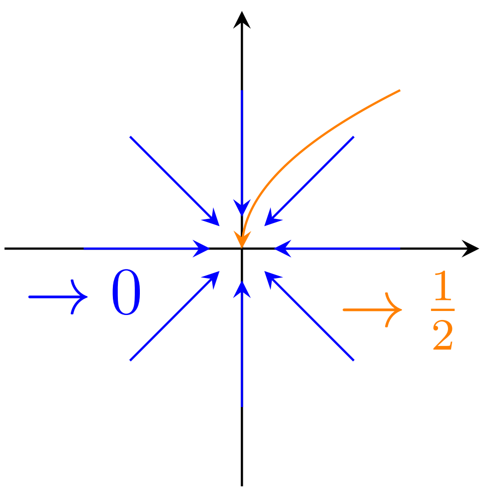
which yields a different result. Since we can approach $(0,0)$ in two different ways and obtain different results, it means that the limit does not exist.




A next idea would be to test all possible directions,

$$\lim_{t\to 0} f(\alpha t, \beta t),$$

with $\alpha, \beta \in \mathbb{R}$ not both zero (thus covering all lines of equation $\beta x - \alpha y = 0$, which are all lines passing through 0). If all the limits along all the lines passing through 0 exist and coincide, can we conclude that the limit exists? <u>The answer is still no</u>! This is because we might obtain a different result when following a trajectory that is not a straight line.




For example, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

then for any $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{t\to 0} f(\alpha t, \beta t) = \lim_{t\to 0} \frac{\alpha\beta^2 t^3}{\alpha^2 t^2 + \beta^4 t^4}.$$

If $\alpha = 0$, then $\beta \neq 0$ and we obtain 0. Otherwise,

$$\lim_{t\to 0} f(\alpha t, \beta t) = \lim_{t\to 0} \frac{\alpha\beta^2 t}{\alpha^2 + \beta^4 t^2} = \frac{0}{\alpha + 0} = 0.$$

We obtain 0 in all directions. However,

$$\lim_{t\to 0} f(t^2, t) = \lim_{t\to 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

Again, this means that the limit does not exist.

The above example shows that if the iterated limits exist and are the same, this does not necessarily imply that the limit of the function exists. However, the converse is true, as the following proposition illustrates.

**Proposition 2.2** (Permutation of limits)**.** *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be a function such that* $\lim_{(x,y)\to(a,b)} f(x,y) = l$. *Suppose, moreover, that for every* $x \in \mathbb{R}$, *the limit* $\lim_{y\to b} f(x,y)$ *exists and that for every* $y \in \mathbb{R}$, *the limit* $\lim_{x\to a} f(x,y)$ *exists. Then,*

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = l$$

**Theorem 2.1** (Squeeze Theorem - Théorème des gendarmes)**.** *Let* $E \subseteq \mathbb{R}^n$, *and functions* $f, g, h : E \to \mathbb{R}$ *be defined on a neighborhood of* $x_0 \in \mathbb{R}^n$. *If*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = l$$

*and there exists* $\alpha > 0$ *such that for all* $\mathbf{x} \in E$,

$$0 < \|\mathbf{x} - \mathbf{x}_0\| < \alpha \implies f(\mathbf{x}) \leqslant h(\mathbf{x}) \leqslant g(\mathbf{x})$$

*then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}) = l.$$

**Example 2.12.** Let us demonstrate that the limit of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy\ln(|x| + |y|) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is zero as $(x,y)$ approaches $(0,0)$ (see Fig. 2.9). On one hand, for every element $(x,y)$ in $\mathbb{R}^2$ satisfying $0 < \sqrt{x^2+y^2} < 1$:

$$0 \leqslant |f(x,y)| = |xy\ln(|x|+|y|)| \leqslant |(|x|+|y|)\ln(|x|+|y|)|.$$

On the other hand, since $\lim_{t\to 0+} t\ln t = 0$ and $\lim_{(x,y)\to(0,0)}(|x|+|y|) = 0$, it follows that:

$$\lim_{(x,y)\to(0,0)} |(|x|+|y|)\ln(|x|+|y|)| = 0.$$

Therefore, using the Squeeze Theorem, we obtain that

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$$

which implies that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

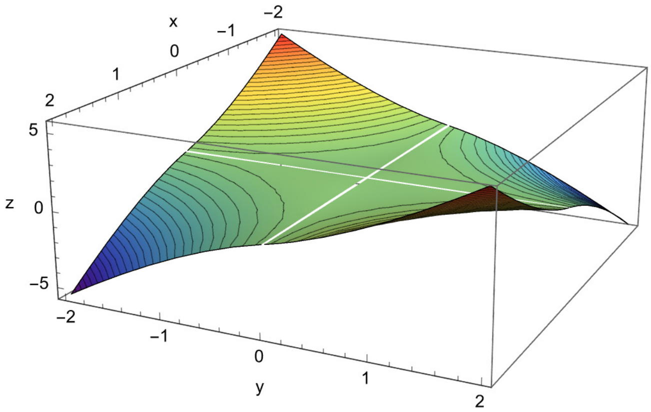



Figure 2.9: Graph of the function $f(x,y) = xy\ln(|x|+|y|)$.

## Limits of functions in two variables

**Theorem 2.2** (Squeeze Theorem in Polar Coordinates)**.** *Let $D \subseteq \mathbb{R}^2$, $(\tilde{x},\tilde{y}) \in \mathbb{R}^2$, $f : D \to \mathbb{R}$ be defined in the neighborhood of $(\tilde{x},\tilde{y})$ and $l \in \mathbb{R}$. Then,*

$$\lim_{(x,y)\to(\tilde{x},\tilde{y})} f(x,y) = l$$

*if and only if there exists $\delta > 0$ and a function $\psi\colon (0,\delta) \to \mathbb{R}$ such that*

*(i)* $\lim_{r\to 0^+} \psi(r) = 0$

*(ii)* $\forall\theta \in [0,2\pi)$ *we have* $|f(\tilde{x}+r\cos\theta, \tilde{y}+r\sin\theta) - l| \leqslant \psi(r)$

**Example 2.13.** Let $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) = x\cos\left(\frac{1}{x^2+y^2}\right).$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

Step 1: Testing the directions. We compute

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0^+} r\cos\theta\cos\left(\frac{1}{r^2\cos^2\theta + r^2\sin^2\theta}\right)$$

$$= \lim_{r\to 0^+} r\cos\theta\cos\left(\frac{1}{r^2}\right).$$

Since

$$-1 \leqslant \cos\left(\frac{1}{r^2}\right) \leqslant 1,$$

it follows that

$$= 0 \cdot \cos\theta = 0.$$

Thus, $l = 0$ is independent of $\theta$.

Step 2: Applying the criterion. We have

$$|f(r\cos\theta, r\sin\theta) - l| = \left| r\cos\theta\cos\left(\frac{1}{r^2\cos^2\theta + r^2\sin^2\theta}\right) - 0\right|$$

$$= \left| r\cos\theta\cos\left(\frac{1}{r^2}\right)\right|.$$

Since

$$|\cos\theta| \leqslant 1,$$

we get

$$\leqslant \left| r\cos\left(\frac{1}{r^2}\right)\right| \to 0.$$

By setting $\psi(r) = \left| r\cos\left(\frac{1}{r^2}\right)\right|$, we have

$$\lim_{r\to 0^+} \psi(r) = 0.$$

Moreover, for all $\theta \in [0, 2\pi]$,

$$|f(r\cos\theta, r\sin\theta) - l| \leqslant \psi(r).$$

Thus, the criterion is satisfied, and we conclude that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

**Example 2.14.** Let $f : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{2|x|}{x^2 + |x| + y^2}.$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x, y).$$

Step 1: Testing the directions. We compute

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0^+} \frac{2r|\cos\theta|}{r^2\cos^2\theta + r|\cos\theta| + r^2\sin^2\theta}$$

$$= \lim_{r\to 0^+} \frac{2r|\cos\theta|}{r^2 + r|\cos\theta|}$$

$$= \lim_{r\to 0^+} \frac{2|\cos\theta|}{r + |\cos\theta|}$$

$$= \begin{cases} 0 & \text{if } \cos\theta = 0 \\ 2 & \text{if } \cos\theta \neq 0 \end{cases}$$

Thus, if we choose $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, we obtain

$$\lim_{r\to 0^+} f(r\cos\theta_1, r\sin\theta_1) = 2$$

$$\lim_{r\to 0^+} f(r\cos\theta_2, r\sin\theta_2) = 0,$$

and therefore, the limit $\lim_{r\to 0^+} f(x, y)$ does not exist.

**Example 2.15.** Let $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{x}{8 + \cos\left(\frac{1}{x^3+y^3}\right)}$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x, y).$$

We use polar coordinates.

Step 1: Directional Test. We compute

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0^+} \frac{r\cos\theta}{8 + \cos\left(\frac{1}{r^3(\cos^3\theta - \sin^3\theta)}\right)}$$

At this stage, note that since $\cos(...) \geqslant -1$, we have

$$8 + \cos\left(\frac{1}{r^3(\cos^3\theta - \sin^3\theta)}\right) \geqslant 7.$$

Since the numerator tends to 0, it follows that

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0,$$

which does not depend on $\theta$, so we choose $l = 0$.

Step 2: Criterion. We have

$$|f(r\cos\theta, r\sin\theta) - l| = \frac{r|\cos\theta|}{8 + \cos\left(\frac{1}{r^3(\cos^3\theta - \sin^3\theta)}\right)} \leqslant \frac{1}{7}r^3.$$

Choosing $\psi(r) = \frac{1}{7}r^3$, we see that $\psi$ satisfies the hypotheses of the squeeze theorem, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

As a side note, this function is a perfect example where the squeeze theorem in Cartesian coordinates works well. We have

$$|f(x,y) - l| \leqslant \frac{|x|}{7}.$$

Choosing $\varphi(x) = \frac{|x|}{7}$, we obtain the desired result.

**Example 2.16.** Now, consider $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 y}{x^2 + y^4}.$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

We switch to polar coordinates.

Step 1: Directional Test. We compute

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0^+} \frac{r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^4\sin^4\theta}$$

$$= \lim_{r\to 0^+} \frac{r\cos^2\theta\sin\theta}{\cos^2\theta + r^2\sin^4\theta}.$$

If $\cos\theta = 0$, the limit is 0 since the numerator is zero. If $\cos\theta \neq 0$, we obtain

$$\frac{0}{\cos^2\theta} = 0.$$

Thus,

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0,$$

so we choose $l = 0$.

Step 2: Criterion. We have

$$|f(r\cos\theta, r\sin\theta) - l| = \frac{r\cos^2\theta|\sin\theta|}{\cos^2\theta + r^2\sin^4\theta}.$$

We estimate the denominator from below. In such cases, we usually use the fact that

$r^2\sin^4\theta \geqslant 0$. This is a finer estimate than using $\cos^2\theta \geqslant 0$, since as $r \to 0$, the term $r^2\sin^4\theta$ vanishes. If we were to use $\cos^2\theta \geqslant 0$ directly, we might introduce an unnecessary error of order 1, for example, when $\theta = 0$.

Thus,

$$|f(r\cos\theta, r\sin\theta) - l| = \frac{r\cos^2\theta|\sin\theta|}{\cos^2\theta + r^2\sin^4\theta} \leqslant \frac{r\cos^2\theta|\sin\theta|}{\cos^2\theta} = r|\sin\theta| \leqslant r.$$

Taking $\psi(r) = r$, we see that the criterion is satisfied, and

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

**Example 2.17.** Let $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Let's discuss the limit

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

We switch to polar coordinates.

Step 1: Directional Test. We compute

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0^+} \frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta}$$

$$= \lim_{r\to 0^+} \frac{r\cos\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}.$$

If $\cos\theta = 0$, the limit is 0 since the numerator is zero. If $\cos\theta \neq 0$, we obtain

$$\frac{0}{\cos^2\theta} = 0.$$

Thus,

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0,$$

so we choose $l = 0$.

Step 2: Criterion. We have

$$|f(r\cos\theta, r\sin\theta) - l| = \frac{r|\cos\theta|\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}$$

$$\leqslant \frac{r|\cos\theta|\sin^2\theta}{\cos^2\theta}$$

$$= \frac{r\sin^2\theta}{|\cos\theta|}.$$

Here, we used the same estimate as before:

$$\cos^2\theta + r^2\sin^4\theta \geqslant \cos^2\theta.$$

However, we run into a problem: the expression in $\theta$, $\frac{\sin^2\theta}{|\cos\theta|}$, is unbounded.

If we instead use another lower bound for the denominator,

$$\cos^2\theta + r^2\sin^4\theta \geqslant r^2\sin^4\theta,$$

we arrive at

$$|f(r\cos\theta, r\sin\theta) - l| \leqslant \frac{|\cos\theta|}{r\sin^2\theta}.$$

However, this expression is even worse. Not only is $\frac{|\cos\theta|}{\sin^2\theta}$ unbounded, but if $\cos\theta \neq 0$, we obtain a term of the form $r^{-1}$, which diverges to infinity instead of converging to zero.

Step 3: Finding a Different Approach to $(0,0)$ That Yields a Different Result.

When dealing with a denominator containing different powers of $x$ and $y$, a good approach is to examine paths of the form $(t^\alpha, t^\beta)$ and choose $\alpha$ and $\beta$ so that the powers of $x$ and $y$ in the denominator match.

In this case, we want the power of $x^2 = t^{2\alpha}$ to match that of $y^4 = t^{4\beta}$. Setting $\beta = 1$ and $\alpha = 2$, we obtain $2\alpha = 4\beta = 4$. Then,

$$\lim_{t\to 0} f(t^2, t) = \lim_{t\to 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

Since this result differs from the directional test, the limit does not exist.